Fleeting Orders

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We study a dynamic limit order market with a finite number of strategic liquidity suppliers who post limit orders. Their limit orders are hit by either news (i.e. informed) traders or noise traders. We show that repeatedly playing a mixed strategy equilibrium of a certain static game is a subgame perfect equilibrium with fleeting orders and flickering quotes. Furthermore, regardless of the distributions of the liquidation value and noise trade quantity, we always find a sequence of equilibria in mixed strategies such that the resulting random supply schedule converges in mean square, as the number of liquidity suppliers increases to infinity, to the deterministic competitive supply function.

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An oft noted feature of today's equity markets is that quoting and quickly canceling are common and frequent events. Nanex describes times in which quote rates exceed 75,000 quotes per second. In one case, Protective Life Corp., which typically trades a few hundred times a day, had 21,000 messages in a 10 second interval in which there was only one transaction.[1] Rapid cancellation of quotes

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[1] www.nanex.net

is often associated with high frequency trading (HFT). In fact, according to the Security and Exchange Commission (SEC) 2010 Concept Release on Equity Market Structure "the submission of numerous orders that are cancelled shortly after submission" is one of the characteristics of HFT. But this is not just a recent phenomenon. We have evidence of flickering quotes, though in a less extreme form, going back to 1999, well before machine trading. Hasbrouck and Saar (2007) document that in 1999 data, on the Island ECN, more than a quarter of quotes were canceled within two seconds.[2]

The rapid cancelation of quotes does not appear to coincide with the competitive equilibrium in Glosten (1994), which predicts a well-behaved supply curve that responds to transactions and other new information. It seems rather implausible to think new information was, in 1999, coming in on a second by second basis, or in 2012, on a millisecond by millisecond basis. Moreover, Hasbrouck (2012) shows that the volatility of quote changes at fifty millisecond intervals is nearly five times what would be predicted by thirty-four minute quote change volatility-volatility that is more likely to represent information arrival. It is thus fairly clear that the flickering of quotes is not due to the arrival of information.

Gaming or even fraudulent behavior by HFT has been proposed as a rationale for flickering quotes. For example, some see excessive quote activity, or quote stuffing, as an attack on the Consolidated Quote System, causing the reporting of quotes to fall behind the reporting of trades.

This paper explores the possibility that limit-order traders manage their undercutting exposure by rapidly canceling their quotes and replacing them with new randomly chosen ones.[3] Given these random choices, it is easy to see why we should see the quotes flicker. Once a constellation of quotes is revealed, a trader will want to revise hers. But everyone knows that everyone else will want to change and they are back to picking another price at random. Thus, quote revi-

[2] In 2000 we thought two seconds was a short period of time.

[3] The notion of random prices is well understood in economic theory. For example Varian (1980) uses the same reasoning to explain the price heterogeneity in sales ads.

sions occur frequently even though trade is sporadic. In a sense, traders mitigate their undercutting risk by "undoing" transparency. Thus, our model supports the notion that flickering quotes are not necessarily a part of a nefarious plan to manipulate the market, but rather the way the liquidity provision game is played. This is our first contribution.

It is standard in market microstructure models of price determination with private information to assume that the liquidity supplied in an electronic limit order book is characterized by a certain zero-profit condition (see Glosten (1994)). The argument presented there is that this is the limit, as the number of players gets large, of the equilibria of games between liquidity suppliers. This is formalized for some environments by Biais et al. (2000, 2013) and Back and Baruch (2013). In particular the latter two make clear that the standard pure strategy equilibrium, in which liquidity suppliers provide supply schedules, may not exist. Furthermore, for a common microstructure model in which there are noise traders and informed traders arriving randomly to the market, Dennert (1993) shows that one equilibrium in mixed strategies does not converge to the competitive equilibrium. Quite the contrary. In his setting, as the number of liquidity suppliers gets large, all the submitted offer quotes pile up at the upper end of the allowable set of prices. Our second contribution is the result that, quite generally, there does exist a sequence of mixed strategy equilibria that converges to the competitive LOB equilibrium in a setting with noise and informed traders.

What we show is that, for the class of market microstructure models with informed and noise trade, it is easy to find a symmetric mixed strategy equilibrium.[4] If there are n competing liquidity suppliers, an equilibrium involves each of them picking a price at random and quoting $\frac{1}{n-1}$ of the maximum noise trade quantity (for ease of exposition, normalize this largest trade to one). The random prices are i.i.d. and hence the number of shares offered at a price p or lower is a bino-

[4] According to a Wall Street Journal article published on 11/20/2012 "exchanges have had to institute mandated cord lengths from the main exchange server... one high-frequency firm's computer a foot from the exchange server and one across the room will have the same cord length so as neither is seen as having a split-second advantage," contributing to the symmetry of the game.

mial random variable (the number of liquidity suppliers that happened to choose a price lower than p) divided by $n-1$. This is approximately the sample mean of n Bernoulli random variables. As n gets large, this converges to a constant function of p.[5] We prove that this limit is the number of shares offered at p or lower in the competitive equilibrium. An implication of this convergence result is that, in the limit, each individual's quote flickers, but the aggregate limit order book is forecastable.

The paper is set out as follows. Section 1 lays out the dynamic game and presents an example that illustrates existence. We also obtain a rather novel result from the example that as the probability of informed trade goes to zero, the stage game expected profit goes to zero but the expected profit in the dynamic game is positive. Section 2 analyzes the game presented in Dennert (1993) and shows, in contrast, that there is a sequence of equilibria converging to the competitive limit order book. Section 3 shows the difficulty of obtaining a pure strategy equilibrium. Section 4 proves, for the general news/noise trader model the existence of a mixed strategy equilibrium that converges to the competitive limit order book. The equilibrium described in Section 4 is a "no rents" equilibrium. Section 5 illustrates convergence to the competitive of positive rents equilibria. Section 6 concludes the paper.

I. Continuous Time Market and Flickering Quotes

We consider a market for a single risky asset and risk free asset with interest rate set to zero. Orders arrive at the market instantaneously and trade is reported instantaneously. However, quotes are disseminated only when trade takes place or after Δt units of time, whichever comes first. Any real number is a feasible price; i.e. the tick size is zero.

The market is organized as a pure limit order book market with the usual price-time priority: An incoming marketable buy (resp. sell) order walks up

[5] The proof is a little more difficult, since as n changes, the mixing distribution changes.

(resp. down) the book picking off outstanding limit orders at their limit prices.[6] If at the last transaction price there is more than one outstanding limit order, then a tie occurs. Ties are resolved using the time precedence rule; i.e. the limit order that was submitted earlier gets to transact.

Information relevant to the value of the risky asset is released to the public at a random time $\tilde{\tau} \sim exp(\theta)$. Following the public announcement, the value of the asset is realized and the asset liquidates.[7] The liquidation value, denoted by $\tilde{v}$, is drawn from either a continuous or a discrete random variable. We denote by F_V and $\bar{v}$ its distribution function and least upper bound, respectively.[8]

There are three types of traders in the market: noise traders, news traders, and limit-order traders. The noise traders submit market orders.[9] The cumulative order flow of the noise traders is a compound Poisson process, $\tilde{z}_t$, with symmetric jumps

$$\tilde{z}_t = \sum_{m=1}^{\tilde{N}_t} \tilde{\epsilon}_m \tilde{q}_m$$

where $\tilde{N}_t$ is a standard Poisson process with intensity β, $\tilde{\epsilon}_m \in \{-1, 1\}$ indicates whether the m-th order is a buy or sell order (we assume equal probabilities), and $\tilde{q}_m$ is the order size of the m-th order, drawn from a common distribution F_Q with a least upper bound of one.[10] We use the notation $\tilde{q}$ for a generic order size. Accordingly, the symbol q denotes a positive number. We assume that $\tilde{\tau}$, $\tilde{v}$, and $\tilde{z}_t$ are independent.

News traders learn $\tilde{v}$ at $\tilde{\tau}$. An instant before the limit-order traders can refresh

[6] A marketable order is an order that can be executed upon submission. Any buy (resp. sell) limit order with a limit price greater than then the ask price (resp. smaller than the bid price) is marketable. In particular, market orders are marketable.

[7] The assumption that the asset liquidates following the announcement is a convenient way to wrap up the model.

[8] If $\tilde{v}$ is unbounded, then $\bar{v}$ equals infinity.

[9] All the results in this paper hold if we assume instead that noise traders submit buy and sell limit orders with prices equal the upper and lower least value of $\tilde{v}$, respectively.

[10] The latter assumption is a normalization that allows us to seamlessly move from the equilibrium mixing distribution to the deterministic competitive supply function.

their quotes, news traders pick off all stale bids and offers. I.e.; news traders submit buy or sell orders with unbounded size and limit price $\tilde{v}$.

Having specified the exogenous strategies of the noise and news traders, we turn our attention to the group of limit-order traders. This group consists of n strategic risk neutral traders. The i-th limit-order trader has, at time t, a collection of outstanding orders that contains all buy and sell limit orders that were submitted in the past and were not executed or canceled prior to time t. We denote this collection by b_i^t. The trader may send a message to the exchange at time t. A message contains instructions to add new limit orders and/or cancel existing ones. After the exchange executes the instructions at time t, the updated collection of limit orders is b_i^{t+}.

We sort the orders in b_i^t according to their price, and summarize the result in a non-decreasing price schedule $P_i^t : R \setminus \{0\} \to R^+$ with the interpretation that $P_i^t(q)$ and $P_i^t(-q)$ are the prices of the q-th unit that the i-th trader offers and bids, respectively.

Analogously, we can express the sorted collection of orders in a non-decreasing function $S_i^t : R^+ \to R \setminus \{0\}$. In its positive range, $S_i^t(p)$ is the number of shares the i-th trader offers at prices smaller or equal to p, and in its negative range, $S_i^t(p)$ is the number of shares the i-th trader bids at prices greater or equal than p. We let $S_{-i}^t = \sum_{j \neq i} S_j^t$. Informationally, P_i^t and S_i^t are equivalent, and we use them interchangeably.[11]

We look for a stationary equilibrium with fleeting orders in which time precedence plays no role. In this equilibrium, traders cancel their limit orders immediately after quotes are disseminated by the exchange and replace them with new ones. The fresh orders are viewed as random by other market participants and to emphasize this uncertainty we write $\tilde{S}_{-i}^t$. The equilibrium is stationary in the

[11] Formally, we take supply function S to be right continuous left limit (resp. left continuous right limit) in its positive (resp. negative) range. We take price schedule P to be left continuous right limit (resp. right continuous left limit) in its positive (resp. negative) domain. In its positive domain $P(q) = \inf\{p : S(p) \geq q\}$, and in its negative domain $P(-q) = \sup\{p : S(p) \leq -q\}$. We can reconstruct S from P in a similar manner.

sense that the distributions of $\tilde{S}^t_{-i}$, is independent of time.

To keep our exposition succinct, in the following we compute the i-th trader's best response using an artificial payoff function that is not smaller than the one prescribed by the game. More specifically, we assume that ties are always broken in favor of the i-th trader. Since ties may occur only when noise traders trade, this is an advantage. Our approach is valid if ties occur with zero probability. In that case, the optimal strategy for the artificial payoff function is also the best response. We emphasize that we only alter the payoff function, but we do not restrict the i-th trader's strategy.

Suppose a market buy order of size $\tilde{q}$ arrives at time $t < \tilde{\tau}$. The order walks up the book picking off limit orders until the order is filled up. If the i-th trader offers his/her q-th unit at p, then the trader sells this unit (i.e. sells at least q) if and only if $S^t_{-i}(p-)$, the number of shares other traders offer at prices *strictly* smaller than p, plus q is still smaller than the size of the incoming order $\tilde{q}$; i.e. $q + \tilde{S}^t_{-i}(p-) \leq \tilde{q}$. The payoff is

$$\pi_0(P^t_i, \tilde{S}^t_{-i}, \tilde{q}, \tilde{v}) = \int_0^\infty I_{\{q+\tilde{S}^t_{-i}(P_i(q)-)\leq\tilde{q}\}}(P^t_i(q) - \tilde{v})dq$$

Analogously, we compute $\pi_0(P^t_i, \tilde{S}^t_{-i}, -\tilde{q}, \tilde{v})$, the payoff when the incoming market order is a sell order of size $\tilde{q}$. Therefore, the i-th trader's payoff at time $t < \tilde{\tau}$ is $\pi_0(P^t_i, \tilde{S}^t_{-i}, d\tilde{z}_t, \tilde{v})$, which is typically zero except at those times when a noise order arrives; i.e. when $d\tilde{z}_t \neq 0$. We integrate out the random variables $\tilde{v}$, the sign and the size of the jump, and $\tilde{S}^t_{-i}$, and get the expected payoff when trading with noise traders:

$$\bar{\pi}_0(P^t_i) \equiv E\left[\pi_0(P^t_i, \tilde{S}^t_{-i}, d\tilde{z}_t, \tilde{v})\middle| I_{\{\Delta\tilde{N}_t>0\}}\right]$$

Note that because $\tilde{S}^t_{-i}$ has a stationary distribution, the functional $\bar{\pi}_0$ is independent of time.

News traders pick off all stale limit orders; i.e. news traders submit limit orders with unbounded size. The payoff at time $\tilde{\tau}$ is:

$$\pi_1(P_i^\tau, \tilde{v}) = \int_0^\infty (P_i^\tau(q) - \tilde{v}) I_{\{\tilde{v} \geq P_i^\tau(q)\}} dq + \int_0^\infty (\tilde{v} - P_i^\tau(-q)) I_{\{\tilde{v} \leq P_i^\tau(-q)\}} dq$$

Note that again the functional π_1 is time independent. We integrate $\tilde{v}$ out to get the expected payoff at time τ:

$$\bar{\pi}_1(P_i^\tau) = E[\pi_1(P_i^\tau, \tilde{v}) | \tilde{\tau} = \tau]$$

The expected payoff of the i-th trader at time s is

$$\begin{aligned}
\Pi_i(s) &= E \int_s^{\tilde{\tau}} \pi_0(P_i^t, \tilde{S}_{-i}, d\tilde{z}_t, \tilde{v}) d\tilde{N}_t + \pi_1(P_i^{\tilde{\tau}}, \tilde{v}) \\
&= E \int_s^{\tilde{\tau}} \bar{\pi}_0(P_i^t)\beta \; dt + \bar{\pi}_1(P_i^{\tilde{\tau}}) \\
&= \int_s^\infty e^{-\theta(t-s)} \left[\bar{\pi}_0(P_i^t)\beta + \theta \bar{\pi}_1(P_i^t)\right] \; dt \\
&= \int_s^\infty (\beta + \theta) e^{-\theta(t-s)} \left[(1-\mu)\bar{\pi}_0(P_i^t) + \mu \bar{\pi}_1(P_i^t)\right] \; dt
\end{aligned} \tag{1}$$

where for the second equality, we integrate out all random variables except $\tilde{\tau}$, which we integrate out in the third equality. The forth equality is a change of variable, where $\mu \equiv \theta/(\theta + \beta)$.

The profit flow is

$$\bar{\pi}(P_i) = [(1-\mu)\bar{\pi}_0(P_i) + \mu \bar{\pi}_1(P_i)] \tag{2}$$

and it is time independent.

To sum up, if $\tilde{S}_{-i}$ is stationary, and the allocation rules favor the i-th trader, then the expected payoff flow is history independent. To find a subgame perfect equilibrium we analyze the following stage game.

The Stage Game: In the stage game the book is initially empty. Next, the limit-order traders simultaneously submit price schedules. With probability μ a noise trader trades, the size of the order is $\tilde{q}$, and with probability $(1-\mu)$ news traders trade. Allocations are determined by the price priority rule, and ties are broken using an unspecified random mechanism.

If a trader knows that there is a positive probability that a tie will occur at p', then the trader can offer/bid the same number of shares, $\Delta S_i(p')$, at a slightly better price. Moreover, because all traders would like to have ties broken in their favor, the infinitesimal undercutting reasoning implies that in any equilibrium of the stage game ties occur with zero probability.[12]

Let $\tilde{P}_i$ be a symmetric mixed-strategy equilibrium of the stage game. Consider the dynamic game. Assume that each of the $j \neq i$ limit-order traders uses a stationary fleeting order strategy; i.e. by the time the exchange disseminates quotes, the traders have replaced their quotes with new ones drawn from the same mixing distribution of the stage game equilibrium. A standard pointwise maximization (for each t maximize the integrand of (1)) implies that each P_i in the support of $\tilde{P}_i$ belongs to the argmax set of (1). Moreover, it is also optimal for the i-th trader to cancel outstanding quotes each time the exchange disseminates quotes, and replace them with new random quotes drawn from the mixing distribution of the stage game equilibrium. Because, in the stage game, ties occur with probability zero, the argmax of (1) is also the set of best responses. We conclude that the mixed strategy equilibrium of the stage game is a Nash equilibrium with fleeting orders of the dynamic game. Because history does not play any role in this equilibrium, the equilibrium is subgame perfect.

[12] A formal proof that ties occur with probability zero can be constructed along the lines of Proposition 3 in Varian (1980).

If we denote by π^* the value of (2) when using the equilibrium strategies, then the equilibrium expected expected payoff in the dynamic game is

$$\Pi^* = \int_s^\infty (\beta + \theta) e^{-\theta(t-s)} \pi^* \, dt = \frac{\pi^*}{\mu} \tag{3}$$

The following example illustrates our theory. The example shows that limit-order traders may randomize both the size and the limit price of their orders. The example also demonstrates that even for marginal levels of adverse selection, the profits in the dynamic game may be bounded away from zero.

Example: Let $n = 2$, $\tilde{v}$ be either 0 or 1 with equal probabilities, $\tilde{q}$ be either 1/2 of a lot, with probability 3/4, or a lot, with probability 1/4.[13]

We start with the offer side of the book and hence we compute $\pi^*/2$. We postulate that an equilibrium in mixed strategies exists in which each of the limit-order traders offers half a lot at a random price, with support $(ask, 1)$, and a second half either at the same price (with probability l) or at one (with probability $1 - l$). Thus, the unknowns are the constants ask, l, π^*, and the mixing distribution function M_a. For $p \in (ask, 1)$, we have

$$\frac{\pi^*}{2} = \frac{\mu}{2}(p-1) + \frac{1-\mu}{2}(p-0.5)\big(1 - M_a(p) + M_a(p)0.25(1-l)\big) \tag{4}$$

$$0 \geq \frac{\mu}{2}(p-1) + \frac{1-\mu}{2}(p-0.5)(1 - M_a(p))0.25, \quad \text{with equality if } l > 0 \tag{5}$$

Examining the right side of (4): with probability μ, a news event occurs and with probability 0.5 the news trader buys in which case the limit order at p will lose $(1-p)$. With probability $\frac{(1-\mu)}{2}$ a noise trader buyer arrives and the profit will be $(p-0.5)$. A limit order at p will transact with a noise trader if either the other limit order is placed higher than p, which occurs with probability $(1 - M_a(p))$; or the other limit order is at p or lower, that limit order only offers half, and the noise trader buys one. All of this happens with probability $M_a(p)(1-l)0.25$.

[13] In Section V we extend this example to arbitrary n.

Equation (4) states that the expected payoff associated with offering the first half is independent of the offering price. Thus, the limit-order trader is willing to randomize the limit order price.

The right hand side of (5) is the expected profit when a second half is also offered at p (i.e. if $l > 0$). The expected profit must be zero if the second half is offered at a price smaller than one, since the right hand side of (5) is zero at $p = 1$. Because we focus on the offer side, the expected profit is half the total profit; i.e., the left hand side of (4) is $\pi^*/2$.

To compute the equilibrium, we start with the guess $l = 0$. Thus, equation (4) is reduced to

$$\pi^* = \mu(p-1) + (1-\mu)(p-0.5)(1 - M_a(p) + M_a(p)0.25)$$

At $p = 1$, $M_a(p) = 1$, and we get $\pi^* = (1-\mu)0.5 \cdot 0.25$. We plug π^* back into (4) and solve for $M_a(p)$. We then find ask by solving $M_a(ask) = 0$. We verify that as long as $\mu \geq 1/9$, (5) holds. However, when $\mu < 1/9$, it is "profitable" to offer an additional half at prices smaller than one; i.e. the right hand side of (5) is strictly positive.

To find the equilibrium when $\mu < 1/9$, we guess $l > 0$, and use (5) to solve for $M_a(p)$. We then find the lower support of M_a by solving $M_a(ask) = 0$. At $p = ask$, (4) gives us

$$\pi^* = \mu(ask - 1) + (1-\mu)(ask - 0.5)$$

We then solve (4) for l and verify that l is a constant. We note that our initial guess $l > 0$ holds only if $\mu < 1/9$.

When pasting the two solutions, we get M_a, ask, l and π^* that are continuous in μ. They are given by

$$M_a(p) = \frac{p - ask}{(1 - ask)(2p - 1)}, \quad ask \leq p < 1$$

where the lower support is

$$ask = \begin{cases} \frac{1+7\mu}{2+6\mu} & \mu \leq 1/9 \\ \frac{5+3\mu}{8} & \mu \geq 1/9 \end{cases}$$

The probability that a second half is offered at the same price as the first one is

$$l = \begin{cases} \frac{1-9\mu}{1+3\mu} & \mu \leq 1/9 \\ 0 & \mu \geq 1/9 \end{cases}$$

and the expected profit is

$$\pi^* = \begin{cases} \frac{3\mu(1-\mu)}{6\mu+2} & \mu \leq 1/9 \\ \frac{1-\mu}{8} & \mu \geq 1/9 \end{cases}$$

Symmetrically, each limit-order trader bids half at a random price p distributed $M_b(p) = 1 - M_a(1-p)$ and bids the second half at the same price with probability l and at zero with probability $1 - l$.

LEMMA 1: *An equilibrium with the above mentioned properties exists.*

The proof of the lemma is in the appendix. Even though both traders offer shares at 1 (with positive probability, and when $\mu \geq 1/9$ with probability one), the tie breaking rule is not required because the noise trader's order is always executed at prices smaller than one. In addition, the mixing distribution is continuous, and therefore ties occur with probability zero.

We note that as μ goes to zero, the expected profit in the stage game goes to zero. This means that the profit flow in the dynamic game goes to zero. However, as μ goes to zero, the number of transactions before a news event gets arbitrarily large, and from (3) the aggregate profit in the dynamic game is strictly positive:

$$\lim_{\mu \to 0} \pi^*/\mu = \lim_{\mu \to 0} \frac{3\mu(1-\mu)}{\mu(6\mu+2)} = \frac{3}{2}$$

This concludes the example.

What if the equilibrium of the stage game is in pure strategies? If S_{-i} is continuous so ties never occur, we can implement the equilibrium in the dynamic game in exactly the same way. However, limit orders are now forecastable. So there is no need to cancel orders just to replace them with identical orders. Thus, if there is an equilibrium in pure strategies, then traders send messages to the exchange only after trade takes place. That is, traders only replenish executed orders. In Section V we provide an example in which we can find an equilibrium in pure strategies.

II. Convergence

We saw that if the stage game has a mixed strategy equilibrium, then there is an equilibrium with fleeting orders in which quotes are random and short lived. This calls into question the assertion that the competitive equilibrium is a viable description of quotes with a large number of limit-order traders. We will show that as their number increases, the total equilibrium random supply function converges, in mean square, to the competitive supply function.

Because the stage game equilibrium is played repeatedly, we can focus on the stage game. In addition, thanks to the symmetry of the game, we can examine the offer side of the book separately from the bid side. Therefore, for the remaining of the paper, we analyze only the offer side of the stage game.

We define the functions $v(p) = E[\tilde{v}|\tilde{v} > p]$, and

$$G(p) = \frac{\mu(v(p) - p)(1 - F_V(p))}{0.5(1 - \mu)(p - E\tilde{v})} \tag{6}$$

We need the following technical result.

LEMMA 2: *The equation $G(p) = 1$ has a solution $p_c > E\tilde{v}$. In the interval*

$(p_c, \bar{v})$, the function $G(p)$ is continuous, strictly decreasing, and

$$G(p_c) = 1$$
$$\lim_{p \uparrow \bar{v}} G(p) = 0$$

Consequently, in the interval $(0, 1)$, the inverse function G^{-1} is also strictly decreasing.

THEOREM 1: *Assume $\tilde{q} \equiv 1$. The stage game has a symmetric equilibrium in mixed strategies in which each limit-order trader offers $1/(n-1)$ at a random price with distribution function*

$$M_n(p) = (1 - G(p))^{1/(n-1)}, \; p \in (p_c, \bar{v}) \tag{7}$$

In this equilibrium, the limit-order traders earn zero profit.

PROOF:

From Lemma 2 it follows that the mixing distribution is well defined. Suppose other traders follow the strategy stated in the theorem, and consider the problem of the i-th trader. Let $\tilde{p}_j$ be j-th trader's random offering price. Clearly to offer shares at prices strictly smaller than p_c is suboptimal because the trader can offer the shares at p_c and still be ahead of all other offers in the book.

The expected profit associated with offering the q-th unit at $p \geq p_c$, and as long as $q \leq 1/(n-1)$, is

$$\begin{aligned}
&\mu E[(p - \tilde{v}) I_{\{p < \tilde{v}\}}] + (1 - \mu) E[I_{\{q + \tilde{S}_{-i}(p) < 1\}}](p - E\tilde{v}) \\
&= \mu(p - v(p))(1 - F_V(p)) + \frac{(1-\mu)}{2}(1 - Prob(p > \max_{j \neq i} \tilde{p}_j))(p - E[\tilde{v}]) \\
&= \mu(p - v(p))(1 - F_V(p)) + \frac{(1-\mu)}{2}(1 - M_n(p)^{n-1})(p - E[\tilde{v}]) \\
&= 0
\end{aligned} \tag{8}$$

where for the last equality we use the definition of the mixing distribution. If

the trader were to offer strictly more than $1/(n-1)$, then to trade with a noise trader, the offering price for the "higher units" has to be better than at least two other random offering prices. Because the probability of undercutting two random prices is strictly smaller than the probability of undercutting one, it follows that the payoff of higher units has to be negative,

We conclude that the i-th trader is indifferent at which price, in the support of $M_n(p)$, to offer each of the the first $1/(n-1)$ units. In particular, it is optimal to offer a block of $1/(n-1)$ at a random price.

The equilibrium in Theorem 1, however, is not unique. Dennert (1993) looks at a special case of Theorem 1 in which the liquidation value is either -1 or 1, and reports that in equilibrium each limit-order trader offers one at a random price.[14] In this equilibrium, to gain from trade (i.e. trade with a noise trader) a limit-order trader has to post the best offer in the book. As a result, the chances of trading with a noise trader decrease with n, and the mixing distribution shifts to the right as n increases. In particular, the sequence of equilibria does not converge to the competitive equilibrium.

In contrast with the result in Dennert (1993), in the equilibrium in Theorem 1, to gain from trade, a limit-order trader has to undercut only one of the other limit-order traders. To see that the equilibrium in Theorem 1 converges to the competitive equilibrium, let $\tilde{S}_n(p)$ denote the total number of shares offered in equilibrium at prices smaller or equal to p, and let $S_c(p)$ denote the supply function in the competitive equilibrium.

THEOREM 2: *As n goes to infinity, the equilibrium in Theorem 1 converges, in mean square, to the competitive equilibrium; i.e.*

$$E(\tilde{S}_n(p) - S_c(p))^2 \to 0$$

[14] Dennert (1993) models a dealer market, where the active trader shops for the best available price. In limit order markets, offers are already ranked from best to worse by the exchange. Thus, the equilibrium in Dennert can be implemented in a limit order market. More generally, any equilibrium in a dealer market in which dealers do not offer quantity discounts can be implemented in a limit order market.

PROOF:

When $\tilde{q} \equiv 1$, the competitive supply function is

$$S_c(p) = I_{\{p \geq p_c\}}$$

In the mixed strategy equilibrium, the total supply of shares is

$$\tilde{S}_n(p) = \frac{1}{n-1} \sum_{i=1}^{n} I_{\{\tilde{p}_i \leq p\}} \tag{9}$$

Because $(n-1)\tilde{S}_n(p) \sim B(n, M_n(p))$, we have

$$E\tilde{S}_n(p) = \frac{n}{n-1} M_n(p) \xrightarrow[n\to\infty]{} I_{\{p \geq p_c\}} = S_c(p)$$

and

$$Var(\tilde{S}_n(p)) = \frac{n}{(n-1)^2} M_n(p)(1 - M_n(p)) < \frac{n}{4(n-1)^2} \xrightarrow[n\to\infty]{} 0$$

Thus,

$$E(\tilde{S}_n(p) - S_c(p))^2 = Var(\tilde{S}_n(p)) + \Big(E\tilde{S}_n(p) - S_c(p))\Big)^2 \xrightarrow[n\to\infty]{} 0$$

The convergence of the equilibrium in Theorem 1 cannot be uniform because the competitive supply function is discontinuous at the ask price. The convergence is illustrated in Figure 1. Even with a huge number of limit-order traders ($n = 1,000$), the depth at the ask price suffices for only about 80% of the noise order. The remaining 20% of the order executes at a dramatically higher price than the competitive. In the following, we consider continuous economies in which the competitive supply function is smooth and the convergence is uniform (Corollary 1 in Section IV and in Figure 2.)

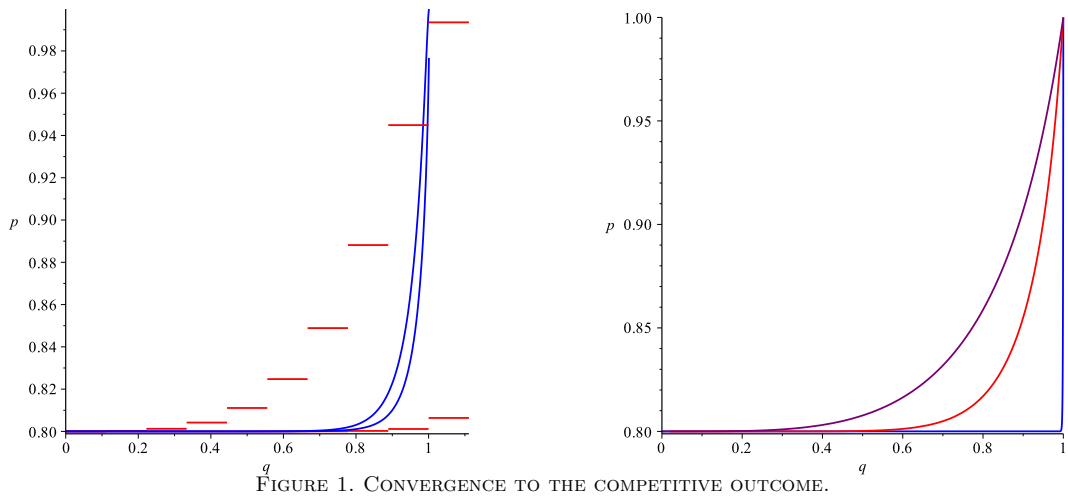

FIGURE 1. CONVERGENCE TO THE COMPETITIVE OUTCOME.

Note: In both figures, the q axis is the competitive equilibrium (i.e. $S_c(0.8) = 1$). In the left figure, the two step functions define the 95% confidence band when $n = 10$; E.g. with probability 0.95 the asking price for the first, second and third $1/(n-1)$ units are virtually the competitive price 0.8. On the other hand, the last fraction of a noise order of size one is executed, with 0.95 probability, anywhere between 0.801 and 0.942. The narrow band, in the same figure, is the 95% confidence band when $n = 1,000$. The right figure shows the mixing distributions when $n = 5$ (the upper curve), $n = 10$ (the middle curve), and $n = 1,000$ (the lowest curve). In both figures, the liquidation value is either zero or one with equal probabilities, the noise order size is deterministic and equals to one, and $\mu = 0.6$.

III. The Continuous Economy

It is common in the literature to assume that the random variables can take on any real value. This abstraction sometimes make the analysis tractable. We therefore further assume that $\tilde{q}$ is a continuous random variable with support $(0, 1)$.

The competitive equilibrium is given implicitly by

$$F_Q(S_c(p)) = 1 - G(p), \qquad p \in (p_c, \bar{v}) \tag{10}$$

where $G(p)$ is defined in (6), $p_c = G^{-1}(1)$ is the competitive ask price, and $\bar{v} = G^{-1}(0)$.

We follow Back and Baruch (2013), and conjecture that there is an equilibrium

in which S_{-i} is continuous. We define the profitability function

$$u(p,q) = \mu(p - v(p))(1 - F_V(p)) + \frac{1-\mu}{2}\,(p - E\tilde{v})\,(1 - F_Q(q)) \tag{11}$$

If S_{-i} is continuous, then the objective of the ith trader is to choose a non-decreasing price schedule P that maximizes

$$\int_0^\infty u(P(q), q + S_{-i}(P(q)))\, dq \tag{12}$$

The objective (12) can be maximized pointwise; i.e., for each $q \geq 0$, maximize the function $p \to u(p, q + S_{-i}(p))$. The f.o.c. is

$$\frac{\partial}{\partial p} u(p, q + S_{-i}(p))\Big|_{p=P^*(q)} = 0 \tag{13}$$

We can now use the symmetric equilibrium condition, namely, $S_{-i} = (n-1)S^*$ to derive an o.d.e. that the total supply function function, S_n, satisfies at prices greater than the ask price:

$$u_p(p, S_n(p)) + \frac{(n-1)}{n} S_n'(p) u_q(p, S_n(p)) = 0 \tag{14}$$

The solution of the o.d.e. is strictly increasing (because $u_p > 0$ and $u_q < 0$), and hence the individual supply function $S_n(p)/n$ is feasible. Moreover, the sequence of solutions converges to the competitive equilibrium supply function as n goes to infinity.[15]

The pointwise optimization we carried above is valid if $p \to u(p, q + S_{-i}(p))$ is quasi-concave. It is not obvious that this should be the case. In fact, if we assume that $\tilde{q}$ is a standard uniform random variable, we can readily see that the

[15]The competitive supply function satisfies $u(p, S_c(p)) = 0$, and therefore, expressed in terms of a differential equation, S_c is the solution of the o.d.e. $u_p(p, S_c(p)) + S_c'(p) u_q(p, S_c(p)) = 0$.

pointwise objective is quasi-convex! Indeed,

$$\frac{\partial^2}{\partial p \partial q} u(p, q + S_{-i}(p)) = \frac{\mu - 1}{2} < 0$$

which implies that for every p, the function $q \to \frac{\partial}{\partial p} u(p, q + S_{-i}(p))$ is strictly decreasing. Thus, for every p there is a value, call it $S^*(p)$, such that for all $q > 0$,

$$\frac{\partial}{\partial p} u(p, q + S_{-i}(p)) \begin{cases} < 0, \text{if } q > S^*(p) \\ > 0, \text{if } q < S^*(p) \end{cases}$$

A priori, $S^*(p)$ may be zero or infinity, however from (13), it follows that $S^*(p)$ must be the inverse of P^*. Thus,

$$\frac{\partial}{\partial p} u(p, q + S_{-i}(p)) \begin{cases} < 0, \text{if } P^*(q) > p \\ > 0, \text{if } P^*(q) < p \end{cases}$$

We conclude that the objective function of the pointwise maximization, $p \to u(p, q + S_{-i}(p))$, is first decreasing and than increasing and hence it is quasi-convex. Thus, when $\tilde{q}$ is uniformly distributed, S^* is the not an an equilibrium individual supply function. We will see in Section V a different distributional assumption for which S^* is an equilibrium.

IV. Convergence and the Continuous Economy

In this section we show the existence of a sequence of Nash equilibria with a random aggregate supply function that converges to the the competitive supply function in the continuous economy given by (10). To construct the equilibria, we discretize the order size that noise traders use. That is, in the nth economy there are n limit-order traders, and the order size of the noise trader is a lattice

random variable, $\tilde{q}_n$, with support

$$\{1/(n-1), 2/(n-1), \ldots, (n-1)/(n-1)\}$$

The demand $\tilde{q}_n$ is related to the demand in the continuous economy via

$$\tilde{q}_n \equiv \frac{\lceil \tilde{q}(n-1) \rceil}{n-1},$$

where $\lceil x \rceil$ is the smallest integer larger than x. In particular,

$$Prob(\tilde{q}_n \leq j/(n-1)) = F_Q(j/(n-1)) \tag{15}$$

Note that even though we use a lattice model, the feasible strategies are general and we do not restrict the limit-order traders to discrete orders. However, in the following, we prove the existence of a symmetric equilibrium in which each limit-order trader offers a block of $1/(n-1)$ at a single random price, $\tilde{p}_i$. That is,

$$\tilde{S}_n(p) = \sum_{i=1}^{n} \frac{1}{n-1} I_{\{\tilde{p}_i \leq p\}}$$

and thanks to the symmetry, $(n-1)\tilde{S}_n(p) \sim Bin(n, M_n(p))$, where $M_n(p)$ is the common mixing distribution. We have the following

LEMMA 3: *Assume* $(n-1)\tilde{S}_{-i}(p) \sim Bin(n, M_n(p))$, *and let*

$$K(p) = Prob(\tilde{q}_n > \tilde{S}_{-i}(p)) \tag{16}$$

Then

$$K(p) = 1 - EF_Q(\tilde{S}_{-i}(p))$$

PROOF:

The definition of the lattice variable $\tilde{q}_n$ implies that for any point in its support, say $j/(n-1)$ for some integer $j \leq (n-1)$, we have $\tilde{q}_n \leq j/(n-1)$ if and only if

$\tilde{q} \leq j/(n-1)$. Because $(n-1)\tilde{S}_{-i}(p)$ is a binomial random variable, and hence an integer random variable, $\tilde{S}_{-i}(p)$ takes values only in the support of the $\tilde{q}_n$. Hence,

$$\begin{aligned} &Prob(\tilde{q}_n \leq \tilde{S}_{-i}(p)) \\ &\qquad = E[E[I_{\{\tilde{q}_n \leq \tilde{S}_{-i}(p)\}}|\tilde{S}_{-i}]] = E[E[I_{\{\tilde{q} \leq \tilde{S}_{-i}(p)\}}|\tilde{S}_{-i}]] = E[F_Q(\tilde{S}_{-i}(p))] \end{aligned}$$

THEOREM 3: *In the lattice model there exists a symmetric Nash equilibrium in which each limit-order trader offers* $1/(n-1)$ *at a random price. The limit-order traders break even, and the distribution function of the random price is given implicitly by*

$$M_n(p) = h(G(p)),\ p \in (p_c, \bar{v})$$

where G *is defined in* (6)*, and* $h(\cdot)$ *is the inverse of the function*

$$k(h) = 1 - E[F_Q(\tilde{j}/(n-1))],\ \ \tilde{j} \sim Bin(n-1, h)$$

In particular, for every $p \in (p_c, \bar{v})$*, we have* $K(p) = G(p)$

The proof is in the Appendix.

LEMMA 4: *In the lattice equilibrium, we have*

$$E[F_Q(\tilde{S}_{-i})] = F_Q(S_c(p)) \tag{17}$$

where $S_c(p)$ *is the competitive supply function in the continuous economy.*

PROOF:

Outside the support of $M_n(p)$ the identity is obvious. For $p \in (p_c, \bar{v})$, we have

$$1 - F_Q(S_c(p)) = G(p) = K(p) = 1 - E[F_Q(\tilde{S}_{-i})]$$

where the first equality is (10), the second equality is from Theorem 3, and the last equality is (16).

COROLLARY 1: *As we increase n, the equilibrium mixing distribution, M_n, converges uniformly to the competitive supply function, $S_c(p)$.*

The proof of the corollary is involved because the transformations between the mixing distribution, the expected random supply, and the competitive supply function are all implicit. The proof is deferred to the appendix. That said, if we assume that $\tilde{q}$ is a standard uniform random variable, then the corollary is immediate. From Lemma 4, we have

$$S_c(p) = E[\tilde{S}_{-i}] = M_n(p)$$

In this example the mixing distribution is exactly the competitive supply function, and in particular the mixing distribution is independent of n. The strategy itself depends on n, because the number of units offered is $1/(n-1)$. Endowed with Corollary 1, the convergence result is immediate.

THEOREM 4: *As n goes to infinity, the equilibrium in lattice economy converges, in mean square, to the competitive equilibrium.*

PROOF:

Because $(n-1)\tilde{S}_n(p) \sim Bin(n, M_n(p))$, we have

$$\lim_{n\to\infty} E[\tilde{S}_n(p)] = \lim_{n\to\infty} \frac{n}{n-1} M_n(p) = S_c(p)$$

where the last equality is the Corollary. Additionally,

$$Var(\tilde{S}_n(p)) = \frac{n}{(n-1)^2} M_n(p)(1 - M_n(p)) < \frac{n}{4(n-1)^2} \xrightarrow[n\to\infty]{} 0$$

and therefore

$$E[(\tilde{S}_n(p) - S_c(p))^2] = Var(\tilde{S}_n(p)) + \Big(E[\tilde{S}_n(p)] - S_c(p))\Big)^2 \xrightarrow[n\to\infty]{} 0$$

V. Economic Rents

The lattice equilibrium in Theorem 3 is a workhorse model: without making distributional assumptions about $\tilde{v}$ and $\tilde{q}$, the equilibrium converges to the competitive. However, in this equilibrium the strategic limit-order traders break even. This type of equilibrium is easy to work with because once we have verified that to offer $1/(n-1)$ has zero expected profits, it follows immediately that one cannot gain by offering even more units. If we were to look for equilibrium with positive expected profit, then we have to carefully check whether it is optimal or not to offer additional units.

In this section, we present an example of equilibrium with positive expected profit, and the equilibrium converges to the competitive. Interestingly, the example we consider can also be dealt with using the technology developed in Back and Baruch (2013). In the example, the liquidation value, $\tilde{v}$, is either zero or one, with equal probabilities, so (6) reduces to

$$G(p) = \frac{\mu(1-p)}{(1-\mu)(p-0.5)}$$

and $p_c = 0.5(1+\mu)$. Also, we assume that $\tilde{q}$ has a triangular distribution with strictly decreasing density; i.e. $1 - F_Q(q) = (1-q)^2$.

Thus, the competitive equilibrium in this example is

$$S_c(p) = 1 - \sqrt{G(p)}, \qquad p \in (p_c, 1)$$

The no-rents equilibrium mixing distribution in Theorem 3 is given by

$$M_n(p) = \begin{cases} 1 - G(p) & n = 2 \\ 1 + \frac{1}{2(n-2)} - \sqrt{\frac{1}{4(n-2)^2} + \frac{(n-1)G(p)}{(n-2)}} & n > 2 \end{cases}$$

and by Theorem 4 this equilibrium converges to the competitive equilibrium.

A. *Mixed Strategy, Positive Rents*

To construct a mixed strategy equilibrium with rents that converges to the competitive, we assume that the order size is a lattice random variable, $\tilde{q}_n$, with support

$$\{1/n, 2/n, \ldots, n/n\}$$

and distribution $Prob(\tilde{q}_n \leq j/n) = F_Q(j/n)$. Note that the case $n = 2$ was studied in Section III. In the notation of the $n = 2$ case, we posit $l = 0$ and we search for an equilibrium in which each of the limit-order traders offers $1/n$ at a random price, with support $(ask_n, 1)$ and yet another $1/n$ at one. Given our results for $n = 2$, we suspect we need to impose a lower bound on μ. Thus, for $p \in (ask_n, 1)$, (4) and (5) become

$$\frac{\pi^*}{2} = \frac{\mu}{2}(p-1) + \frac{1-\mu}{2}(p-0.5)E(1-\tilde{S}_{-i}(p))^2 \tag{18}$$

$$0 \geq \frac{\mu}{2}(p-1) + \frac{1-\mu}{2}(p-0.5)E(1-\tilde{S}_{-i}(p)-1/n)^2 \tag{19}$$

We get π^* by evaluating (18) at $p = 1$:

$$\pi^* = (1-\mu)(1-0.5)\frac{1}{n^2}$$

After noting that $\tilde{S}_{-i}(p)$ is $1/n$ times a binomially distributed random variable with parameters $n-1$ and the mixing distribution at p, $M_n(p)$, we get a quadratic

equation in $M_n(p)$. The solution is given by:

$$M_n(p) = \frac{\frac{(n-1)(2n-1)}{n^2} - \sqrt{\frac{(n-1)^2(2n-1)^2}{n^4} - 4\left[1 - G(p) - \frac{0.5}{n^2(p-0.5)}\right]\frac{(n-2)(n-1)}{n^2}}}{2\frac{(n-1)(n-2)}{n^2}}$$

To find the ask price, we solve $M_n(ask_n) = 0$ and get

$$ask_n = 0.5(1+\mu) + \frac{(1-\mu)}{2n^2}$$

To satisfy the inequality (19) for all p between ask_n and 1, we need to assume

$$\mu \geq \frac{(n-1)}{(n+1)(2n-1)}$$

Obviously, as n gets large, the constraint becomes non-binding. That is, the existence of this type of equilibrium is more likely the greater the number of limit-order providers. One could speculate, based on the $n = 2$ case, that for smaller μ a doubly mixed strategy might be an equilibrium; i.e. traders randomize prices and quantities.

It is easy to see that $M_n(p)$ converges to the competitive supply function. As noted above, the supply at a price p or below is the sample mean of independent Bernoulli trials with success probability $M_n(p)$ and hence, for the reasons given in Section IV, the random supply function converges to the competitive supply function. Figure 2 illustrates the convergence.

B. Pure Strategy, Positive Rents

The profitability function (11) is

$$u(p,q) = \mu(p-1)0.5 + \frac{1-\mu}{2}(p-0.5)(1-q)^2$$

and the solution to the o.d.e. (14) is

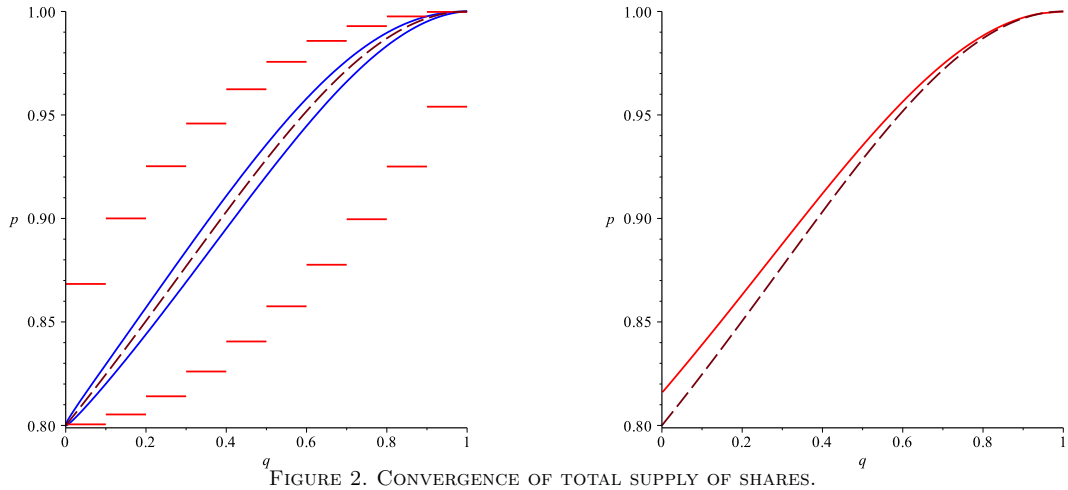

FIGURE 2. CONVERGENCE OF TOTAL SUPPLY OF SHARES.

Note: The left figure corresponds to the equilibrium in mixed strategies with rents. The two step functions define the 95% confidence band when $n = 10$; E.g. with probability 0.95 the asking price for the first 0.1 shares is anywhere between 0.8 and 0.87 while the asking price for the next 0.1 shares is between 0.805 and 0.9. The solid inner band is the 95% confidence band when $n = 1,000$. Finally, the dashed curve corresponds to the competitive equilibrium. The right figure corresponds to the pure strategy equilibrium. The solid line is the total supply curve when $n = 10$, and the dashed curve is the competitive supply function. The case $n = 1,000$ is not shown because it is indistinguishable from the competitive. In both figures, the liquidation value is either zero or one with equal probabilities, the noise order size has the the triangle distribution, and $\mu = 0.6$.

$$S_n(p) = 1 - \sqrt{\frac{\mu}{1-\mu}}\sqrt{(2p-1)^{\frac{-n}{n-1}} - 1}, p \in (ask_n, 1)$$

where ask_n is

$$ask_n = 0.5\left(1 + \mu^{\frac{n-1}{n}}\right)$$

This may or may not be an equilibrium. As with the example in Section III, it is important to check the second order conditions that for all $q > 0$ and at the solution $S_n(p)$:

$$\frac{\partial}{\partial p}u(p, q + S_{-i}(p)) \begin{cases} < 0, \text{if } q < S_n(p)/n \\ > 0, \text{if } q > S_n(p)/n \end{cases}$$

In the case at hand $\frac{\partial}{\partial p}u(p, q + S_{-i}(p))$ is given by:

$$0.5\mu + 0.5(1-\mu)\left(1 - q - \frac{n-1}{n}S_n(p)\right)^2 \\ - \frac{n-1}{n}S_n'(p)(1-\mu)(1 - q - \frac{n-1}{n}S_n(p))(p - .5)$$

Note that this expression is quadratic and convex in q and that $S_n(p)/n$ is one of its zeros. We need to check that $S_n(p)/n$ derived above is its larger root. To do so we check that the derivative of the above with respect to q evaluated at $S_n(p)/n$ is positive:

$$-(1-\mu)\left(1 - q - \frac{n-1}{n}S_n(p)\right) + \frac{n-1}{2n}S_n'(p)(1-\mu)(2p-1) > 0$$

After substitution for $S_n(p)$ and $S_n'(p)$ we note that the derivative above is given by:

$$\frac{\sqrt{\mu(1-\mu)}}{2\sqrt{(2p-1)^{\frac{-n}{n-1}} - 1}}\left[2 - (2p-1)^{-n/(n-1)}\right]$$

This derivative should be positive for all p and that is determined by the expression in square brackets. This expression is increasing in p and positive at $p = 1$. It will be positive for all p if it is positive at ask_n. Noting the expression for ask_n above, the derivative will be positive if and only if $2 - \frac{1}{\mu} > 0$ or $\mu > .5$. We also need to check that the smaller root of the quadratic equation in q is less than zero. Brute force shows that this is true if $\mu > .5$. Thus, the pure strategy equilibrium is as described above as long as there is sufficient adverse selection, namely $\mu > 0.5$. If on the other hand, $\mu < 0.5$ then, if there is a pure strategy equilibrium, it is not as characterized above. Figure 2 contrasts this equilibrium with the equilibrium in mixed strategies.

VI. Conclusion

The contribution of the analysis in this paper is twofold: (1) it provides a strategic foundation for the competitive equilibrium in Glosten and Milgrom (1985) and Glosten (1994), and (2) it shows that short lived orders are an equilibrium outcome. With regard to the second result, the model is "too successful" in the sense that the life span of all orders, in equilibrium, is at most the latency of the data feed. There are, however, two caveats. First, the tick size in the model is zero, and therefore time precedence, in equilibrium, is moot. The combination of a coarse price grid and time precedence should increase the life span of orders. The second caveat is that liquidity in our model is solely provided by a group of traders who hope to profit from the trading game. While this dichotomy is a time honored assumption in the literature, in limit order markets anyone can provide liquidity.[16] Traders who need immediacy can post aggressive limit orders and in so doing drive out of the market our group of liquidity suppliers. Similarly, traders who are only interested in profiting from the trading game may find it profitable to pick off limit orders posted by these liquidity traders.

Despite the above mentioned limitations, the model gives a simple economic rational for fleeting orders: limit-order traders worried about being undercut can effectively hide their quotes by using short lived orders at random prices.

Our analysis finds its most immediate purpose in interpreting the results of Hasbrouck (2012). That paper shows that while message traffic has increased dramatically over the past decade (2001-2011), the volatility of the National best bid and offer has not increased, but the nature of the volatility has. Our two results–convergence and the robustness of mixed strategy equilibria–predict such a result. At the individual level, quotes are entered and cancelled quickly, yet at the aggregate level, and with sufficient competition, the National best bid and offer need not change very much.[17] On the other hand if there are very few

[16] Roşu (2009) presents a model of a limit order market without the group of liquidity suppliers.
[17] See figure 1 which shows that for $n = 10$ the best offer is very close to the competitive price.

competing liquidity suppliers the best bid and offer can be far more volatile. Thus a security can have either stable best bid and offer or rapidly oscillating bids and offers; each case involving a high level of message traffic but different level of competition.

Appendix

Proof of Lemma 1: To verify that we have found an equilibrium, we consider the offer side of the stage game, the bid side is symmetric. We assume the second trader uses the equilibrium strategy, and consider the problem of the first trader. Offering more than one cannot be optimal because noise traders buy at most one. We therefore focus on price schedules $P_1 : [0,1] \to R^+$. We note that shares offered at prices greater than one are never executed because trader 2 offers one at prices smaller or equal to one.

The expected profit on the offer side is

$$\begin{aligned}
&\frac{\mu}{2}\int_0^1 I_{\{P_1(q)\le 1\}}(P_1(q)-1)\,dq + \frac{1-\mu}{1}\int_0^1 Prob(q+\tilde{S}_2(P_i(q)-)\le \tilde{q})(P_1(q)-0.5)\,dq \\
&= \int_0^{1/2} I_{\{P_1(q)\le 1\}}\left[\frac{\mu}{2}(P_1(q)-1)\right. \\
&\qquad\qquad \left. +\frac{1-\mu}{2}\left(1-M_n(P_1(q))+\frac{M_n(P_1(q))(1-l)}{4}\right)(P_1(q)-0.5)\right]dq \quad (I) \\
&\quad + \int_{1/2}^1 I_{\{P_1(q)\le 1\}}\left[\frac{\mu}{2}(P_1(q)-1)+\frac{1-\mu}{2}\cdot\frac{1-M_n(P_1(q))}{4}(P_1(q)-0.5)\right]dq \quad (II)
\end{aligned}$$

The integrand of (I) is $\pi^*/2$ as long as $P_1(q) \in (ask, 1]$, and strictly less otherwise. The integrand of (II) is zero if (i) $P_1(q) = 1$ or (ii) $\mu \le 1/9$ and $P_1(q) \in (ask, 1]$. The integrand of (II) is strictly negative otherwise.

We conclude that it is also optimal for the first trader to offer half at any price in $(ask, 1]$. In particular it is optimal to randomize. Also, there is no harm in offering an additional half at one, and when $\mu \le 1/9$ it is also optimal to offer the additional half at the same price at which the first half is offered. In particular,

it is optimal to randomize and with probability l to offer one at the same price. Thus, we have verified the equilibrium.

Proof of Lemma 2: In the interval $(E\tilde{v}, \bar{v}]$, the function $G(p)$ is continuous. To see this, note that the denominator in (6) continuous. The numerator in (6) is

$$\mu \int_p^{\bar{v}} (v-p) I_{\{p<v\}} dF_V(v)$$

which is continuous in p whether $\tilde{v}$ is a discrete or continuous random variable. Thus, $G(p)$ is continuous in $(E\tilde{v}, \bar{v}]$.

In the interval $(E\tilde{v}, \bar{v}]$, the function $G(p)$ is strictly decreasing. Indeed, the derivative of the numerator is $\mu(F_V(p) - 1) < 0$. The denominator is clearly increasing in p. Hence we conclude that $G(p)$ is strictly decreasing.

Because $\lim_{p \downarrow \bar{v}} G(p) = 0$, and $\lim_{p \downarrow E\tilde{v}} G(p) = \infty$, it follows that a solution to the equation $G(p) = 1$ exists. Because G is strictly decreasing, its inverse is also strictly decreasing.

Proof of Theorem 3: We first show that the mixing distribution in Theorem 3 is well defined. Because F_Q is a distribution, clearly, $k(0) = 1$ and $k(1) = 0$. Also,

$$\begin{aligned} k(h) &= 1 - EF_Q(\tilde{j}/(n-1) = 1 - EE[I_{\{\tilde{q} < \tilde{j}/(n-1)\}} | \tilde{j}] = 1 - EE[I_{\{(n-1)\tilde{q} < \tilde{j}\}} | \tilde{q}] \\ &= EB((n-1)\tilde{q}; n-1, h) \end{aligned}$$

where $B(x; n, h)$ is the distribution function of a binomial random variable with n Bernoulli trials, each with a probability of successes h. Since $B(x; n, h)$ is strictly decreasing with h, it follows that also the expectation, $k(h)$, is strictly decreasing. [18]

[18] Formally, let $j = \lfloor x \rfloor$ be the floor of x, then

$$B(j, n, h) = (n-j)\binom{n}{j} \int_0^{1-h} t^{n-j-1}(1-t)^j dt$$

and hence the probability is strictly decreasing. Informally, when we increase the probability of success in each trial, then the probability of having a total of j or less successes strictly decreases - while this is not true for the probability of having exactly j successes, it is true for the cumulative probability.

From Lemma 2, we know that G is strictly decreasing, and hence $M_n(p) = h(G(p))$ is strictly increasing. To conclude that M_n is a distribution with support $[G^{-1}(1), G^{-1}(0)]$, we verify:

$$M_n(G^{-1}(1)) = h(1) = 0$$

and

$$M_n(G^{-1}(0)) = h(0) = 1$$

Therefore, M_n is a distribution function.

Next, we apply $k(h)$ to both sides of the definition of $M_n(p)$ to get that in $(G^{-1}(p), G^{-1}(0))$, we have $K(p) = G(p)$. More generally,

$$K(p) = \begin{cases} 1 & p < G^{-1}(1) \\ G(p) & G^{-1}(p) \leq p < G^{-1}(0) \\ 0 & G^{-1}(p) \leq p \end{cases}$$

Consider now the problem of the ith trader, assuming all other limit-order traders follow the strategy stated in the theorem. For $q \in (0, 1/(n-1)]$, we have

$$Prob(\tilde{q}_n \geq q + \tilde{S}_{-1}(P_i(q))) = Prob(\tilde{q}_n > \tilde{S}_{-1}(P_i(q))) = K(p)$$

where the second equality is the definition of $K(p)$. Hence, the expected profits associated with the "first" 1/(n-1) units, each unit may be offered at different prices, is

$$\int_0^{1/(n-1)} \mu(P_i(q) - v(P_i(q)))(1 - F_V(P_i(q))) + \frac{1-\mu}{2}\,(P_i(q) - E\tilde{v})\,K(p)\,dq \quad \text{(A1)}$$

We consider the value of the integrand for different p's. For $p < p_c$, the integrand is negative because p_c is the ask price of the competitive equilibrium. For $p > \bar{v}$,

the integrand is zero.

For p in the support of the mixing distribution, we use the definitions of $G(p)$ (equation (6)) and $M_n(p)$ to conclude that the integrand of the objective (A1) is zero. Therefore, the expected gain on the first $1/(n-1)$ unit is at most zero, and exactly zero if the units are offered in the interval of prices $(p_c, \bar{v})$.

We need to show that it is suboptimal to offer more than $1/(n-1)$ units. But this is obvious because the chances that additional units will be picked by the noise traders are strictly smaller than the probability that the first $1/(n-1)$ units are. Since the profitability of the latter is zero, it follows that it is suboptimal to offer more than $1/(n-1)$ units.

We conclude that it is optimal to offer $1/(n-1)$ in the support of M_n and in particular it is optimal to offer the entire block at the same random price with distribution M_n. Finally, as we have seen, the expected profit is zero.

Proof of Corollary 1: The proof is in steps.

Step 1: Given $\delta > 0$ and $\epsilon > 0$, there exists an N, independent of p, such that for all $n > N$, we have $Prob(|\tilde{S}_{-i}(p) - M_n(p)| > \delta) \leq \epsilon/4$. In particular, for every p, $\tilde{S}_{-i}(p) - M_n(p)$ converges to zero in probability.

Indeed, we have $(n-1)\tilde{S}_{-i}(p) \sim B(n-1, M_n(p))$. Take $N > 1/(\epsilon\delta^2)$, then from Chebyshev's Inequality

$$Prob(|\tilde{S}_{-i}(p) - M_n(p)| > \delta)$$
$$\leq \frac{Var(\tilde{S}_{-i}(p))}{\delta^2} = \frac{M_n(p)(1 - M_n(p))}{(n-1)\delta^2} \leq \frac{1}{4n\delta^2} < \frac{\epsilon}{4}$$

Step 2: $EF_Q(\tilde{S}_{-i}(p)) - F_Q(M_n(p))$ uniformly converges to zero. Let $\epsilon > 0$ be given. We need to show that there is an N such that for all $n > N$ we have $\left|EF_Q(\tilde{S}_{-i}(p)) - F_Q(M_n(p))\right| < \epsilon$.

The distribution function F_Q is continuous in the closed interval $[0, 1]$ and hence it is uniformly continuous. Thus, there exists a $\delta > 0$, associated only with ϵ, such that if $|q_1 - q_2| < \delta$, then $|F_Q(q_q) - F_Q(q_2)| < \epsilon/2$.

We take now $N > 1/(\epsilon\delta^2)$ (as in Step 1). Now,

$$\begin{aligned}
&|EF_Q(\tilde{S}_{-i}(p)) - F_Q(M_n(p))| \\
\leq & E|F_Q(\tilde{S}_{-i}(p)) - F_Q(M_n(p))| \\
= & E|F_Q(\tilde{S}_{-i}(p)) - F_Q(M_n(p))|I_{\{|\tilde{S}_{-i}(p)-M_n(p)|\leq\delta\}} \\
& \qquad + E|F_Q(\tilde{S}_{-i}(p)) - F_Q(M_n(p))|I_{\{|\tilde{S}_{-i}(p)-M_n(p)|>\delta\}} \\
\leq & \frac{\epsilon}{2} + 2Prob\left(\left|\tilde{S}_{-i}(p) - M_n(p)\right| > \delta\right) < \epsilon
\end{aligned}$$

Step 3: We use Lemma 4 to replace, in Step 2, $EF_Q(\tilde{S}_{-i}(p))$ with $F_Q(S_c(p))$ and conclude that $F_Q(S_c(p)) - F_Q(M_n(p)) \to 0$ uniformly.

Step 4: We are now ready to show that $S_c(p) - M_n(p) \to 0$ uniformly. In other words, we need to show that given ϵ, there is an N, independent of p, such that

$$|S_c(p) - M_n(p)| < \epsilon$$

The inverse distribution function, $F_Q^{-1}(x)$ is continuous in $[0, 1]$ and hence uniformly continuous. Thus, there is a $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies $|F_Q^{-1}(x_1) - F_Q^{-1}(x_2)| < \epsilon$. From Step 3, we know that there is an N that depends only δ such that for $n > N$, we have

$$|F_Q(S_c(p)) - F_Q(M_n(p))| < \delta$$

Thus, for $n > N$, we also have

$$|S_c(p) - M_n(p)| = |F_Q^{-1}(F_Q(S_c(p))) - F_Q^{-1}(F_Q(M_n(p)))| < \epsilon$$

where the inequality follows from the uniform continuity of F_Q^{-1}. This ends the proof.

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